Advanced Mathematical Models & Applications
Vol.7, No.1, 2022, pp.60-68

A NOTE ON ZAGREB SPECTRA AND ZAGREB LAPLACIAN SPECTRA OF TWO WEIGHTED CORONA NETWORKS

A. Mahanta[1*], Idweep J. Gogoi[2], A. Bharali[2]

[1]Dergaon Kamal Dowerah College, Assam, India
[2]Dibrugarh University, Assam, India

Abstract. The paper studies two corona networks, namely weighted corona networks and weighted edge corona networks in association with the Zagreb matrices which have gained popularity in recent times. Herein, we study the Zagreb spectra in terms of two different corona structures and also study their Laplacian and signless Laplacian Zagreb spectrum.

Keywords: Zagreb matrices, corona network, edge corona network, graph spectra.
AMS Subject Classification: 05C05, 05C07.
Corresponding author: A. Mahanta, Dergaon Kamal Dowerah College, Assam, India,
e-mail: *am02dib@gmail.com*
Received: 05 December 2021; Revised: 29 December 2021; Accepted: 13 January 2022; Published: 19 April 2022.

1 Introduction

The study of networks has caught the eye of not only the mathematical community but also of physicists from all over the globe in the recent years. The mathematical appeal inherent in many real life networks such as social networks, chemical networks and traffic networks has garnered considerable attraction. One such popular graph network is the corona networks. Spectral properties of various types of corona networks in terms of different types of matrices have been studied over the years. For an insight into this field one may look up Barik et al., 2007; Barik & Sahoo, 2007; Chen & Liao, 2017; Cui & Tian, 2012; Doley et al., 2020; Hou & Shiu, 2010; Mahanta et al., 2021; McLeman & McNicholas, 2011; Tahir & Zhang, 2020; Wang & Zhou, 2013; Zhang, 2013. In this work, we will be considering corona and edge corona networks.

Most of the works carried out in this end have been based on unweighted graphs. But when it comes to capturing information of interactions among the adjacent vertices, weighted graphs turn out to be handy. Bearing this is in mind, we propose a weighted version of corona networks and of edge corona networks.

Now we give some important definitions that we have used in our work.

Definition 1. First Zagreb index: *(Gutman, 2013) The first Zagreb index of a graph G is defined as*

$$M_1(G) = \sum_{v\in V} d_v^2 = \sum_{uv\in E} (d_u + d_v).$$

Definition 2. Second Zagreb index: *(Gutman, 2013) The second Zagreb index of a graph G is defined as*

$$M_2(G) = \sum_{uv\in E} d_u d_v.$$

Definition 3. *Let G_1 and G_2 be two simple graphs. Then the Corona product of two graphs (Frucht & Harary, 1970) G_1 and G_2 denoted by $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the i^{th} vertex of G_1 to every vertex in i^{th} copy of G_2.*

Definition 4. *Let G_1 be the initial graph with order n_1 and size m_1, G_2 be the copy graph with order n_2 and size m_2 respectively. The edge corona (Liu et al., 2020) $G_1 \Diamond G_2$ of G_1 and G_2 is obtained by making one copy of G_1 and m_1 copies of G_2, connecting each vertex of the i^{th} copy of G_2 to the two end vertices of the i^{th} edge of G_1.*

Suppose that the initial graph G_1 is a simple unweighted graph and the copy graph G_2 is a weighted graph with each edge having a weight factor r, $0 \leq r \leq 1$. Then we have two special types of weighted corona networks arising out of the above corona networks.

For a clear understanding, let us consider an example where we take $G_1 = K_3$ and $G_2 = K_2$.

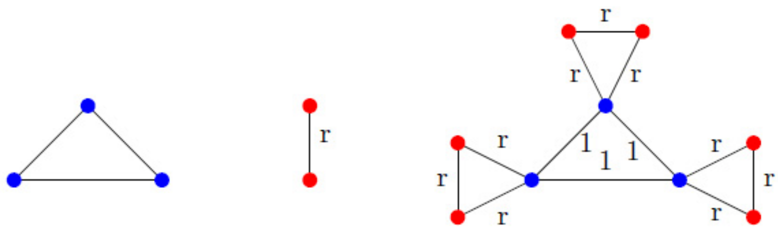

Figure 1: G_1, G_2 and $G_1 \circ G_2$

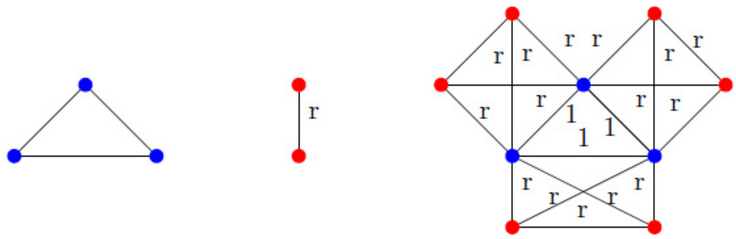

Figure 2: G_1, G_2 and $G_1 \diamond G_2$

The rest of the paper is organised as follows. In section 2, we obtain the *Zagreb* spectra, of the weighted corona network followed by linking the *Zagreb* spectra with its adjacency, Laplacian and signless Laplacian spectra. In section 3, we give the the *Zagreb* spectra, of the weighted edge corona networks. Thereafter we link their *Zagreb* spectra with their adjacency, Laplacian and signless Laplacian spectra. The concluding remarks are made in section 4.

2 Zagreb spectra of the weighted $G_1 \circ G_2$

Theorem 1. *Let G_1 be the d_1-regular graph with order n_1 and size m_1, G_2 be the d_2-regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Then the eigenvalues of $Z_1(G_1 \circ G_2)$ are $(d_1 + rn_2)^2$ with multiplicity n_1 and $r^2(d_2+1)^2$ with multiplicity $n_1 n_2$ respectively.*

Proof. The proof can be easily obtained because

$$Z_1(G_1 \circ G_2) = \begin{bmatrix} (d_1 + rn_2)^2 I_{n_1} & 0_{n_1 n_2} \\ 0_{n_1 n_2} & r^2(d_2+1)^2 I_{n_1 n_2} \end{bmatrix}.$$

□

Theorem 2. *Let G_1 be the d_1-regular graph with order n_1 and size m_1, G_2 be the d_2-regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the second Zagreb spectra of G_1 and G_2 are $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ and $\sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$. Then the spectra of $Z_2(G_1 \circ G_2)$ are as follows*

- $\lambda \in Z_2(G_1 \circ G_2)$ *with multiplicity 1 where λ is the roots of the equation*

$$d_1^2\lambda^2 - \lambda\left(r^2 d_1^2 d_2 (d_2+1)^2 + (d_1+n_2 r)^2 \lambda_i^{(1)}\right) + \left(d_2\lambda_i^{(1)} - n_2 d_1^2\right) r^2 (d_1+n_2 r)^2 (d_2+1)^2 = 0.$$

- $\lambda = \frac{(d_2+1)^2}{d_2^2}\lambda_j^{(2)}$, *for $j = 1, 2, \ldots, n_2 - 1$, with multiplicity n_1.*

Proof. Suppose that J_{n_2} is the row vector with order n_2 whose all elements are 1. In view of the construction of the weighted $G_1 \circ G_2$ it gives the second Zagreb matrix as follows

$$Z_2\left(G_1 \circ G_2\right) = \begin{bmatrix} \frac{(d_1+n_2 r)^2}{d_1^2} Z_2(G_1) & r(d_2+1)(d_1+n_2 r)\left[J_{n_2} \otimes I_{n_1}\right] \\ r(d_2+1)(d_1+n_2 r)[J_{n_2} \otimes I_{n_1}]^T & \frac{(d_2+1)^2}{d_2^2}[Z_2(G_2) \otimes I_{n_1}] \end{bmatrix}.$$

Consider the eigenvector of $Z_2(G_1 \circ G_2)$ corresponding to the eigenvalue λ as $X = [X_1 X_2 \ldots X_{n_2+1}]^T$, where $X_i \in R^{n_1}$. One obtains a part of desired results from the below two cases with $\lambda \neq r^2 d_2 (d_2+1)^2$.

Case 1: Non zero vector X_1.

$$\frac{(d_1+n_2 r)^2}{d_1^2} Z_2(G_1) X_1 + r(d_2+1)(d_1+n_2 r) I_{n_1}\left(X_2 + X_3 + \cdots + X_{n_2+1}\right) = \lambda X_1. \qquad (1)$$

Let $E_i = (\overbrace{0_{n_1} 0_{n_1} \cdots 0_{n_1}}^{i-1} I_{n_1} \overbrace{0_{n_1} 0_{n_1} \cdots 0_{n_1}}^{n_2 - i})$. Then

$$\begin{cases} r(d_2+1)(d_1+n_2 r) I_{n_1}^T X_1 + \frac{(d_2+1)^2}{d_2^2} E_1 [Z_2(G_2) \otimes I_{n_1}][X_2 \cdots X_{n_2+1}]^T = \lambda X_2, \\ r(d_2+1)(d_1+n_2 r) I_{n_1}^T X_1 + \frac{(d_2+1)^2}{d_2^2} E_2 [Z_2(G_2) \otimes I_{n_1}][X_2 \cdots X_{n_2+1}]^T = \lambda X_3, \\ \vdots \\ r(d_2+1)(d_1+n_2 r) I_{n_1}^T X_1 + \frac{(d_2+1)^2}{d_2^2} E_{n_2} [Z_2(G_2) \otimes I_{n_1}][X_2 \cdots X_{n_2+1}]^T = \lambda X_{n_2+1}. \end{cases}$$

Therefore

$$n_2 r (d_2+1)(d_1+n_2 r) I_{n_1}^T X_1 + \frac{(d_2+1)^2}{d_2^2} d_2 r^2 d_2^2 (X_2 + X_3 + \cdots + X_{n_2+1}) = \lambda (X_2 + \cdots + X_{n_2+1})$$

$$\Rightarrow (X_2 + \cdots + X_{n_2+1}) = \frac{n_2 r (d_2+1)(d_1+n_2 r) I_{n_1}^T X_1}{\lambda - r^2 d_2 (d_2+1)^2}.$$

Putting this value in Eq. 1 and simplifying we get

$$\frac{(d_1+n_2 r)^2}{d_1^2} Z_2(G_1) X_1 = \left\{\lambda - \frac{r^2 (d_2+1)^2 (d_1+n_2 r)^2 n_2}{\lambda - r^2 d_2 (d_2+1)^2}\right\} X_1.$$

If the second Zagreb spectra of G_1 is $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ then

$$\frac{(d_1+n_2 r)^2}{d_1^2} \lambda_i^{(1)} = \lambda - \frac{r^2 (d_2+1)^2 (d_1+n_2 r)^2 n_2}{\lambda - r^2 d_2 (d_2+1)^2}.$$

From this we have

$$d_1^2\lambda - \lambda\left(r^2d_1^2d_2(d_2+1)^2 + (d_1+n_2r)^2\lambda_i^{(1)}\right) + \left(d_2\lambda_i^{(1)} - n_2d_1^2\right)r^2(d_1+n_2r)^2(d_2+1)^2 = 0. \quad (2)$$

Case 2: Zero vector X_1.

$$r\,(d_2+1)\,(d_1+n_2r)\,I_{n_1}(X_2+\cdots+X_{n_2+1}) = 0,$$
$$\frac{(d_2+1)^2}{d_2^2}\,[Z_2(G_2)\otimes I_{n_1}]\,[X_2\cdots X_{n_2+1}] = \lambda[X_2\cdots X_{n_2+1}]^T.$$

If $\sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$ then one can straightforward get that

$$\lambda = \frac{(d_2+1)^2}{d_2^2}\lambda_j^{(2)}, j = 1, 2, \ldots, n_2-1, \quad (3)$$

with multiplicity n_1.

Now from Eq. 2 and Eq. 3 we obtained $(n_2-1)n_1 + 2n_1 = n_1 + n_1n_2$ eigenvalues of $Z_2(G_1 \circ G_2)$.

□

Now it will be interesting to compute the Zagreb spectra of weighted corona networks of two different graph structures in association with the adjacency, Laplacian and signless Laplacian spectra of these graphs.

2.1 Zagreb spectra of the weighted $G_1 \circ G_2$ in terms of adjacency spectra

Consider J_{n_2} denote the row vector with order n_2 and all elements are 1. According to the construction of $G_1 \circ G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \circ G_2) = \begin{pmatrix} (d_1+n_2r)^2A(G_1) & r(d_2+1)(d_1+n_2r)[J_{n_2}\otimes I_{n_1}] \\ r(d_2+2)(d_1+d_1n_2r)[J_{n_2}\otimes I_{n_1}]^T & r(d_2+1)^2[A(G_2)\otimes I_{n_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof of the theorem being similar to the one done previously, we have omitted it to avoid repetition.

Theorem 3. *Let G_1 be a d_1-regular graph with order n_1 and size m_1, G_2 the d_2-regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the adjacency spectrum of G_1 and G_2 are $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ and $\sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \circ G_2)$ are as follows.*

- *$\lambda \in Z_2(G_1 \circ G_2)$ with multiplicity 1 where λ is the roots of the equation*
$$\lambda^2 - \lambda\left(r^2d_2(d_2+1)^2 + (d_1+n_2r)^2\lambda_i^{(1)}\right) + \left(d_2\lambda_i^{(1)} - n_2\right)r^2(d_1+n_2r)^2(d_2+1)^2 = 0,$$
$i = 1, 2, \ldots, n_1$.
- *$r(d_2+1)^2\lambda_j^{(2)} \in Z_2(G_1 \circ G_2)$ with multiplicity n_1, $j = 1, 2, \ldots, n_2-1$.*

2.2 Zagreb-Laplacian spectra of the weighted $G_1 \circ G_2$.

Consider J_{n_2} denote the row vector with order n_2 and all elements are 1. According to the construction of $G_1 \circ G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \circ G_2) = \begin{pmatrix} (d_1+n_2r)^2(-L(G_1)+d_1I_{n_1}) & r(d_2+1)(d_1+n_2r)[J_{n_2}\otimes I_{n_1}] \\ r(d_2+2)(d_1+d_1n_2r)[J_{n_2}\otimes B(G_1)]^T & r(d_2+2)^2[(-L(G_2)+rd_2I_{n_2})\otimes I_{n_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof has been omitted to avoid repetition.

Theorem 4. *Let G_1 be a d_1-regular graph with order n_1 and size m_1, G_2 the d_2-regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the Laplacian spectrum of G_1 and G_2 are $l(G_1) = \{\mu_1^{(1)}, \mu_2^{(1)}, \dots, \mu_{n_1}^{(1)}\}$ and $l(G_2) = \{\mu_1^{(2)}, \mu_2^{(2)}, \dots, \mu_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \circ G_2)$ are as follows.*

- *$\mu \in Z_2(G_1 \circ G_2)$ with multiplicity 1 where μ is the roots of the equation*

$$\mu^2 - \mu\left(r^2 d_2(d_2+1)^2 + d_1(d_1+n_2r)^2 - (d_1+n_2r)^2\mu_i^{(1)}\right) +$$

$$+\left(d_1d_2 - n_2 - d^2\mu_i^{(1)}\right) r^2(d_2+1)^2(d_1+n_2r)^2 = 0,\ i = 1, 2, \dots, n_1.$$

- *$r(d_2+1)^2(-\mu_j^{(2)} + rd_2) \in Z_2(G_1 \circ G_2)$ with multiplicity n_1, $j = 1, 2, \dots, n_2 - 1$.*

2.3 Zagreb-Signless Laplacian spectra of the weighted $G_1 \circ G_2$

Consider J_{n_2} denote the row vector with order n_2 and all elements are 1. According to the construction of $G_1 \circ G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \circ G_2) = \begin{pmatrix} (d_1+n_2r)^2(Q(G_1) - d_1I_{n_1}) & r(d_2+1)(d_1+n_2r)[J_{n_2} \otimes I_{n_1}] \\ r(d_2+1)(d_1+n_2r)[J_{n_2} \otimes I_{n_1}]^T & r(d_2+1)^2[(Q(G_2) - rd_2I_{n_2}) \otimes I_{n_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof has been omitted to avoid repetition.

Theorem 5. *Let G_1 be a d_1-regular graph with order n_1 and size m_1, G_2 the d_2-regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the signless Laplacian spectrum of G_1 and G_2 are $s(G_1) = \{\delta_1^{(1)}, \delta_2^{(1)}, \dots, \delta_{n_1}^{(1)}\}$ and $s(G_2) = \{\delta_1^{(2)}, \delta_2^{(2)}, \dots, \delta_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \circ G_2)$ are as follows.*

- *$\delta \in Z_2(G_1 \circ G_2)$ with multiplicity 1 and δ is the root of*

$$\delta^2 - \delta\left(r^2d_2(d_2+1)^2 - d_1(d_1+n_2r)^2 + (d_1+n_2r)^2\delta_i^{(1)}\right) +$$

$$+\left(r^2d^2\delta_i^{(1)} - r^2d_1d_2 - r^2n_2\right)(d_2+1)^2(d_1+n_2r)^2 = 0,\ i = 1, 2, \dots, n_1.$$

- *$r(d_2+1)^2(\delta_j^{(2)} - rd_2) \in Z_2(G_1 \circ G_2)$ with multiplicity n_1, $j = 1, 2, \dots, n_2 - 1$.*

3 Zagreb spectra of the weighted $G_1 \diamond G_2$

Theorem 6. *Let G_1 be the d_1-regular graph with order $n_1$and size m_1, G_2 be the d_2-regular graph with order n_2 and size m_2 respectively. Then the eigenvalues of $Z_1(G_1 \diamond G_2)$ are $(d_1 + rd_1n_2)^2$ with multiplicity n_1 and $r^2\left(d_2+2\right)^2$ with multiplicity m_1n_2 respectively.*

Proof. The proof can be easily obtained because

$$Z_1(G_1 \diamond G_2) = \begin{bmatrix} (d_1+rd_1n_2)^2 I_{n_1} & 0_{m_1n_2} \\ 0_{m_1n_2} & r^2\left(d_2+2\right)^2 I_{m_1n_2} \end{bmatrix}.$$

□

Theorem 7. *Let G_1 be the d_1-regular graph with order $n_1$and size m_1, G_2 be the d_2-regular graph with order n_2 and size m_2 respectively. Assume that the second Zagreb spectra of G_1 and G_2 are $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_{n_1}^{(1)}\}$ and $\sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_{n_2}^{(2)}\}$. Then the spectra of $Z_2(G_1 \diamond G_2)$ are as follows*

- $\lambda_{1,2} = \dfrac{-(d_1+d_1n_2r)^2\ \lambda_j^1 - rd_1^2d_2(d_2+2)^2 \pm \sqrt{\begin{matrix}\left((d_1+d_1n_2r)^2\ \lambda_j^1 + rd_1^2d_2(d_2+2)^2\right)^2 - 4\left(-d_1^2\right)\\ (d_2+2)^2(d_1+d_1n_2r)^2 \\ (-d_2r\ \lambda_j^1 + n_2r^2\ \lambda_j^1 + r^2n_2d_1^3)\end{matrix}}}{-2d_1^2}$, $j = 1, 2, \ldots, n_1$, *with multiplicity 1.*
- $\lambda = \frac{(d_2+2)^2}{d_2^2}\lambda_j^2$ *for* $j = 1, 2, \ldots, n_2 - 1$, *with multiplicity* m_1.
- $\lambda = r^2d_2(d_2+2)^2$ *is also an eigenvalue with multiplicity* $m_1 - n_1$ *(if possible).*

Proof. Suppose that J_{n_2} is the row vector with order n_2 whose all elements are 1. In view of the construction of the weighted $G_1 \diamond G_2$ it gives the second Zagreb matrix as follows

$$Z_2(G_1 \diamond G_2) = \begin{bmatrix} \frac{(d_1+d_1n_2r)^2}{d_1^2}Z_2(G_1) & r(d_2+2)(d_1+d_1n_2r)[J_{n_2} \otimes B(G_1)] \\ r(d_2+2)(d_1+d_1n_2r)[J_{n_2} \otimes B(G_1)]^T & \frac{(d_2+2)^2}{d_2^2}[Z_2(G_2) \otimes I_{m_1}] \end{bmatrix}.$$

Consider the eigenvector of $Z_2(G_1 \diamond G_2)$ corresponding to the eigenvalue λ as $X = [X_1X_2\ldots X_{n_2+1}]^T$, where $X_1 \in R^{n_1}$ and $X_i \in R^{m_1}$ otherwise. One obtains a part of desired results from the below two cases with $\lambda \neq r^2d_2(d_2+2)^2$.

Case 1: Non zero vector X_1.

$$\frac{(d_1+d_1n_2r)^2}{d_1^2}Z_2(G_1)X_1 + r(d_2+2)(d_1+d_1n_2r)B(G_1)(X_2+X_3+\cdots+X_{n_2+1}) = \lambda X_1. \quad (4)$$

Let $E_i = \overbrace{0_{m1}\ldots 0_{m1}}^{i-1} I_{m1} \overbrace{0_{m1}\cdots 0_{m1}}^{n_2-i}$. Then

$$\begin{cases} r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1 + \frac{(d_2+2)^2}{d_2^2}E_1[Z_2(G_2)\otimes I_{m_1}][X_2\cdots X_{n_2+1}]^T = \lambda X_2, \\ r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1 + \frac{(d_2+2)^2}{d_2^2}E_2[Z_2(G_2)\otimes I_{m_1}][X_2\cdots X_{n_2+1}]^T = \lambda X_3, \\ \vdots \\ r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1 + \frac{(d_2+2)^2}{d_2^2}E_{n_2}[Z_2(G_2)\otimes I_{m_1}][X_2\cdots X_{n_2+1}]^T = \lambda X_{n_2+1}. \end{cases}$$

Therefore

$$n_2r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1 + \frac{(d_2+2)^2}{d_2^2}d_2r^2d_2^2(X_2+X_3+\ldots X_{n_2+1}) =$$

$$\lambda(X_2+\cdots+X_{n_2+1}) \Rightarrow (X_2+\cdots+X_{n_2+1}) = \frac{n_2r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1}{\lambda - r^2d_2(d_2+2)^2}.$$

Putting this value in Eq. 4, we get

$$\frac{(d_1+d_1n_2r)^2}{d_1^2}Z(G_1)X_1 + \frac{r(d_2+2)(d_1+d_1n_2r)B(G_1)n_2r(d_2+2)(d_1+d_1n_2r)B(G_1)^TX_1}{\lambda - r^2d_2(d_2+2)^2} = \lambda X_1$$

$$\Rightarrow \left\{\frac{(d_1+d_1n_2r)^2}{d_1^2} + \frac{r^2(d_2+2)^2(d_1+d_1n_2r)^2n_2\cdot\frac{1}{d_1^2}}{\lambda - r^2d_2(d_2+2)^2}\right\}Z_2(G_1)X_1$$

$$\left\{\lambda - \frac{r^2(d_2+2)^2(d_1+d_1n_2r)^2n_2d_1}{\lambda - r^2d_2(d_2+2)^2}\right\}X_1.$$

If the second Zagreb spectra of G_1 is $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ then

$$\left\{\frac{(d_1+d_1n_2r)^2}{d_1^2} + \frac{r^2(d_2+2)^2(d_1+d_1n_2r)^2n_2\cdot\frac{1}{d_1^2}}{\lambda - r^2d_2(d_2+2)^2}\right\}\lambda_i^{(1)} = \lambda - \frac{r^2(d_2+2)^2(d_1+d_1n_2r)^2n_2d_1}{\lambda - r^2d_2(d_2+2)^2}$$

From this we have

$$\begin{aligned} &-d_1^2\lambda^2 + \{(d_1 + d_1 n_2 r)^2 \lambda_j^1 + r^2 d_1^2 d_2 (d_2+2)^2\}\lambda + (d_2+2)^2 (d_1 + d_1 n_2 r)^2 - \\ &-r^2 d_2 \ \lambda_j^1 + r^2 n_2 \ \lambda_j^1 + r^2 n_2 d_1^3 = 0. \end{aligned} \tag{5}$$

Case 2: Zero vector X_1.

$$r\,(d_2+2)\,(d_1 + d_1 n_2 r)\, B\,(G_1)\,(X_2 + .. + X_{n_2+1}) = 0$$
$$\frac{(d_2+2)^2}{d_2^2}\left[Z_2\,(G_2) \otimes I_{m_1}\right][X_2 \ldots .X_{n_2+1}] = \lambda [X_2 \ldots X_{n_2+1}]^T$$

If $\sigma(G_2) = \sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$ then one can straightforward get that

$$\lambda = \frac{(d_2+2)^2}{d_2^2}\lambda_j^{(2)}, j = 1, 2, \ldots, n_2 - 1. \tag{6}$$

with multiplicity m_1

Now from Eq. 5 and Eq. 6 we obtain $(n_2 - 1)m_1 + 2n_1$ eigenvalues of $Z_2(G_1 \diamond G_2)$. Hence $\lambda = r^2 d_2 (d_2+2)^2$ is also an eigenvalue with multiplicity $m_1 - n_1$.

□

Now we compute the Zagreb spectra of weighted edge corona networks of two different graph structures in association with the adjacency, Laplacian and signless Laplacian spectra of the graphs.

3.1 Zagreb spectra of the weighted $G_1 \diamond G_2$ in terms of adjacency spectra

Consider J_{n_2} denote the row vector with order n_2 and all elements are 1. According to the construction of $G_1 \diamond G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \diamond G_2) = \begin{pmatrix} (d_1 + d_1 n_2 r)^2 A(G_1) & r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)] \\ r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)]^T & r(d_2+2)^2[A(G_2) \otimes I_{m_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof has been omitted to avoid repetition.

Theorem 8. *Let G_1 be a d_1 regular graph with order n_1 and size m_1, G_2 the d_2 regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the adjacency spectrum of G_1 and G_2 are $\sigma(G_1) = \{\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{n_1}^{(1)}\}$ and $\sigma(G_2) = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \diamond G_2)$ are as follows.*

- $\frac{(d_1+d_1 n_2 r)^2\lambda_i^{(1)} + r^2 d_2 (d_2+2)^2 \pm \sqrt{((d_1+d_1 n_2 r)^2 \lambda_i^{(1)} + r^2 d_2 (d_2+2)^2)^2 - 4(d_1+d_1 n_2 r)^2 (d_2+2)^2 (r^2 d_2 - r^2 n_2)\lambda_i^{(1)} - r^2 n_2 d_1}}{2} \in$ *$z(G_1 \diamond G_2)$, with multiplicity 1, $i = 1, 2, \ldots, n_1$.*
- *$r(d_2+2)^2 \lambda_j^{(2)} \in Z_2(G_1 \diamond G_2)$ with multiplicity m_1, $j = 1, 2, \ldots, n_2 - 1$.*
- *$r^2 d_2 (d_2+2)^2 \in Z_2(G_1 \diamond G_2)$ with multiplicity $m_1 - n_1$(if possible).*

3.2 Zagreb-Laplacian spectra of the weighted $G_1 \diamond G_2$

Consider J_{n_2} denote the row vector with order n_2 and all elements equal 1. According to the construction of $G_1 \diamond G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \diamond G_2) = \begin{pmatrix} (d_1 + d_1 n_2 r)^2(-L(G_1) + d_1 I_{n_1}) & r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)] \\ r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)]^T & r(d_2+2)^2[(-L(G_2) + r d_2 I_{n_2}) \otimes I_{m_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof has been omitted to avoid repetition.

Theorem 9. *Let G_1 be a d_1 regular graph with order n_1 and size m_1, G_2 the d_2 regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the Laplacian spectrum of G_1 and G_2 are $l(G_1) = \{\mu_1^{(1)}, \mu_2^{(1)}, \dots, \mu_{n_1}^{(1)}\}$ and $l(G_2) = \{\mu_1^{(2)}, \mu_2^{(2)}, \dots, \mu_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \diamond G_2)$ are as follows.*

- $\mu \in Z_2(G_1 \diamond G_2)$ *with multiplicity 1 where* μ *is the roots of the equation* $\mu^2 + \mu[(d_1 + d_1 n_2 r)^2 \mu_i^{(1)} - r^2 d_2(d_2+2)^2 - d_1(d_1 + d_1 n_2 r)^2] +$ $+(d_1 + d_1 n_2 r)^2 (d_2+2)^2 r^2 (-d_2 \mu_i^{(1)} + n_2 \mu_i^{(1)} + d_1 d_2 - 2 d_1 n_2) = 0$, $i = 1, 2, \dots, n_1$.
- $r(d_2+2)^2(-\mu_j^{(2)} + r d_2) \in Z_2(G_1 \diamond G_2)$ *with multiplicity* m_1, $j = 1, 2, \dots, n_2 - 1$.
- $r^2 d_2 (d_2+2)^2 \in Z_2(G_1 \diamond G_2)$ *with multiplicity* $m_1 - n_1$ *(if possible).*

3.3 Zagreb-Signless Laplacian spectra of the weighted $G_1 \diamond G_2$

Consider J_{n_2} denote the row vector with order n_2 and all elements are 1. According to the construction of $G_1 \diamond G_2$, it gives the generalized second Zagreb matrix as follows

$$Z_2(G_1 \diamond G_2) = \begin{pmatrix} (d_1 + d_1 n_2 r)^2 (Q(G_1) - d_1 I_{n_1}) & r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)] \\ r(d_2+2)(d_1 + d_1 n_2 r)[J_{n_2} \otimes B(G_1)]^T & r(d_2+2)^2[(Q(G_2) - r d_2 I_{n_2}) \otimes I_{m_1}] \end{pmatrix}.$$

From this we have the following theorem. The proof has been omitted to avoid repetition.

Theorem 10. *Let G_1 be a d_1 regular graph with order n_1 and size m_1, G_2 the d_2 regular graph with order n_2 and size m_2 each of which has weight factor r, respectively. Assume that the signless Laplacian spectrum of G_1 and G_2 are $s(G_1) = \{\delta_1^{(1)}, \delta_2^{(1)}, \dots, \delta_{n_1}^{(1)}\}$ and $s(G_2) = \{\delta_1^{(2)}, \delta_2^{(2)}, \dots, \delta_{n_2}^{(2)}\}$. Then the generalized second Zagreb spectra of $Z_2(G_1 \diamond G_2)$ are as follows.*

- $\delta \in Z_2(G_1 \diamond G_2)$ *with multiplicity 1 and* δ *is the root of* $\delta^2 + \delta[(d_1 + d_1 n_2 r)^2 (d_1 - \delta_i^{(1)}) - r^2 d_2 (d_2+2)^2] - r^2 (d_1 + d_1 n_2 r)^2 (d_2+2)^2 [d_1 d_2 - d_2 \delta_i^{(1)} + n_2 \delta_i^{(1)}] = 0$, $i = 1, 2, \dots, n_1$.
- $r(d_2+2)^2(\delta_j^{(2)} - r d_2) \in Z_2(G_1 \diamond G_2)$ *with multiplicity* m_1, $j = 1, 2, \dots, n_2 - 1$.
- $r^2 d_2 (d_2+2)^2 \in Z_2(G_1 \diamond G_2)$ *with multiplicity* $m_1 - n_1$ *(if possible).*

4 Conclusion

In this work, we look into the Zagreb spectra of weighted corona and edge corona networks of two different graph structures and an attempt has been made to link their Zagreb spectra with the adjacency spectra, Laplacian spectra and signless Laplacian spectra. As a future scope, it will be interesting to study the Zagreb spectra of such structures for different values of the weight factor r other than $[0, 1]$.

References

Barik, S., Pati, S., & Sarma, B.K. (2007). The Spectrum of the Corona of Two Graphs. *SIAM. J. Discrete Math.*, *21*(1), 47-56.

Barik, S., Sahoo, G. (2007). On the Laplacian spectra of some varients of corona. *Linear Algebra Appl.*, *512*(1), 32-47.

Chen, H.Y., Liao, L. (2017). The normalized Laplacian spectra of the corona of two graphs. *Linear Multilinear Algebra*, *65*(3), 582-592.

Cui, S.Y., Tian, G.X. (2012). The spectrum of the singless Laplacian spectrum of coronae. *Linear Algebra Appl.*, *437*, 1692-1703.

Doley, A., Deka, B. & Bharali, A. (2020). Misbalance degree matrix and related energy of graphs. *Journal of Mathematical and Computational Science*, *10*(3), 436-447.

Gutman, I. (2013). Degree-based topological indices. *Croat. Chem. Acta*, *86*(4), 351-361.

Frucht, R., Harary, F. (1970). On the corona of two graphs. *Aequations Mathematicae*, *4*, 322-325.

Hou, Y., Shiu, W. (2010). The spectrum of the edge corona of two graph. *Electron. J. Linear Algebra*, *20*, 586-594.

Liu, J.B., Zhao, J. & Cai, Z.Q. (2020). On the generalized adjacency, Laplacian and Signless Laplacian spectra of the weighted edge corona networks. *Physica A*, *540*(C), 123073.

Mahanta, A., Gogoi, I.J. & Bharali, A. (2021). A note on the generalized adjacency, Laplacian and signless Laplacian spectra of the weighted edge corona networks. *Physica A*, *581*, 126241.

McLeman, C., McNicholas, E. (2011). Spectra of corona. *Linear Algebra Appl.*, *435*, 998-1007.

Tahir, M.A., Zhang, X.D. (2020). Coronae graphs and their α-eigenvalues. *Bulletin of the Malaysian Mathematical Sciences Society*, *43*, 2911-2927.

Wang, S., Zhou, B. (2013). The signless Laplacian spectra of corona an edge corona of two graphs. *Linear Multilinear Algebra*, *289*(19), 454-465.

Zhang, F. (2005). *The Schur Complement and its Applications.* Springer, New York, NY, USA.